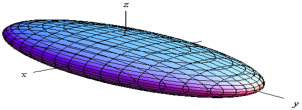
Provably weak instances of Ring-LWE revisited



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[2] Ghent University
[3] Open Security Research

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HEAT

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We revisit the paper

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Currently no threat to Ring-LWE.

1. Learning With Errors (LWE)

The LWE problem (O. Regev, '05): solve a linear system

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Drawback: key size.

- To hide the secret one needs an entire linear system:

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$\uparrow$ $n \log p$ (for b) $\qquad$ $\uparrow$ $mn \log p$ (for the matrix) $\qquad$ $\uparrow$ $n \log p$ (for s)

2. Ring-based LWE

Solution:

- Identify key space

$$\mathbb{F}_p^n \qquad \text{with} \qquad \frac{\mathbb{Z}[x]}{(p, f(x))}$$

for some monic deg n polynomial $f(x) \in \mathbb{Z}[x]$, by viewing

$$(s_0, s_1, \ldots, s_{n-1}) \quad \text{as} \quad s_0 + s_1 x + s_2 x^2 + \cdots + s_{n-1} x^{n-1}.$$

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- Use samples of the form

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- Store $\mathbf{a}(x)$ rather than $A_{\mathbf{a}}$: saves factor n.

2. Ring-based LWE

Example:

- if $f(x) = x^n - 1$, then $A_{\mathbf{a}}$ is the circulant matrix

$$\begin{pmatrix} a_0 & a_{n-1} & \cdots & a_2 & a_1 \\ a_1 & a_0 & \cdots & a_3 & a_2 \\ a_2 & a_1 & \cdots & a_4 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{pmatrix}$$

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- Bad example, because of ...

3. Evaluation-at-1 attack

Potential threat:

- Suppose $f(1) \equiv 0 \bmod p$, then

$$\frac{\mathbb{Z}[x]}{(p, f(x))} \to \mathbb{F}_p : \mathbf{r}(x) \mapsto \mathbf{r}(1) = r_0 + r_1 + \cdots + r_{n-1},$$

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Safety measure: restrict to irreducible $f(x) \in \mathbb{Z}[x]$.

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Direct ring-based analogue of LWE-sample would read

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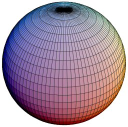

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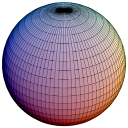

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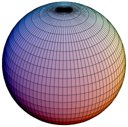

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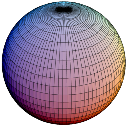

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- Sometimes called Poly-LWE.

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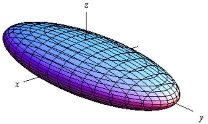

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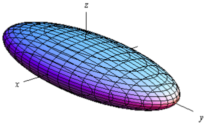

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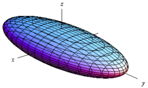
So “on average”, each e_i is scaled up by $\sqrt{\Delta}^{1/n}$ …



- … but remember: skewness.

5. Provably weak instances of Ring-LWE revisited

[ELOS] constructed families of polynomials $f(x)$ that are vulnerable to an evaluation-at-1 attack.

For convenience they picked non-dual secrets:

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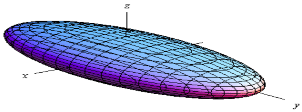
- The factor $\sqrt{\Delta}^{1/n}$ compensates for B^{-1} only "on average".
- In some coordinates B^{-1} could scale down much more.



Compensation factor is insufficient

$\rightsquigarrow$ merely rounding yields exact equations in the secret!

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All instances from [ELOS] suffer from this skewness.

- Example: $f(x) = x^{256} + 8190$, $p = 8191$. $\leftarrow$ note: $f(1) \equiv 0 \bmod p$
- Standard deviations even form a geometric series!
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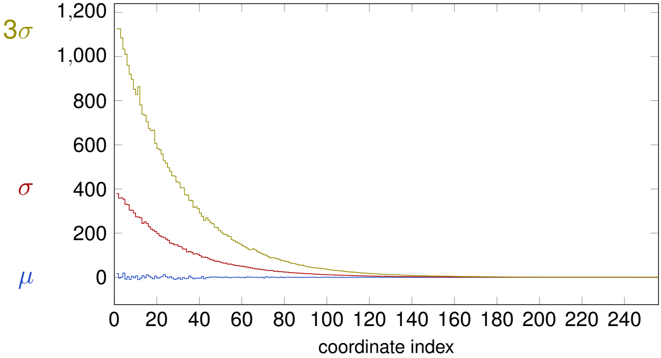

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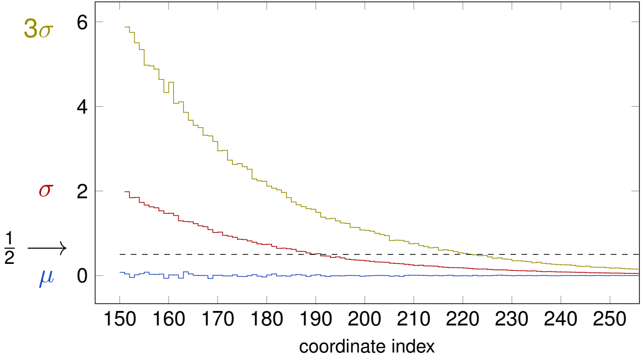

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Similar remarks apply to the other instances from [ELOS].

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- If one does scale the [ELOS] examples sufficiently, then the error coordinates of low index become uniform.
- The cyclotomic case seems naturally protected against geometric growth.