Faster homomorphic comparison operations for BGV and BFV

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Privacy Enhancing Technologies Symposium

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Our data is kept in the cloud

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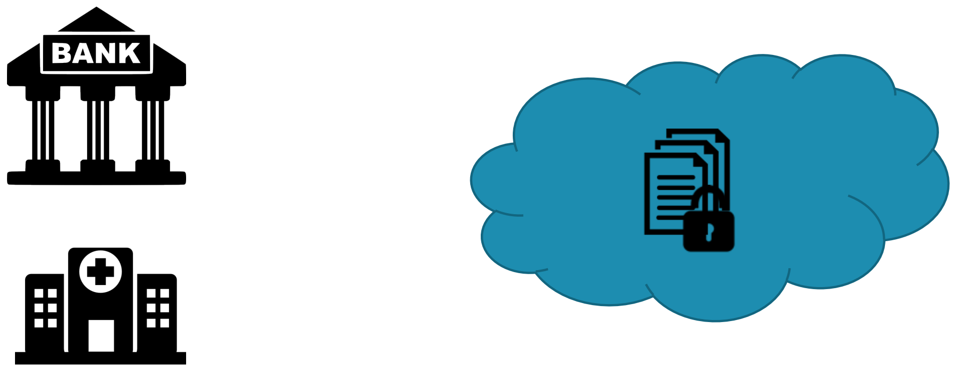


How to work with encrypted data in the cloud?

Security

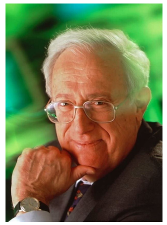
Functionality

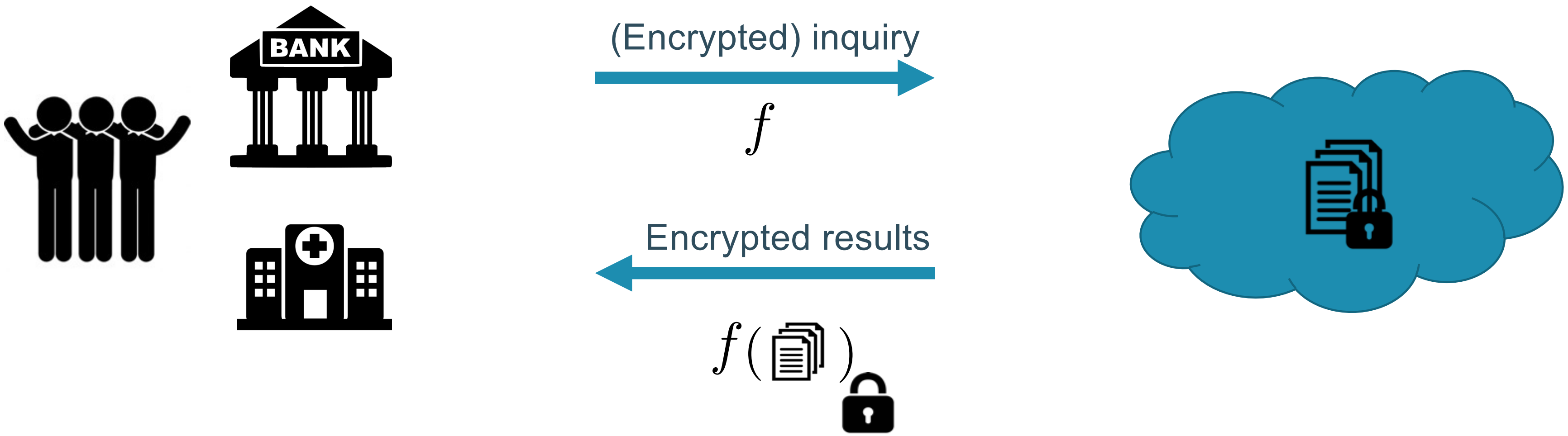
Homomorphic encryption

1978

ON DATA BANKS AND PRIVACY HOMOMORPHISMS

Ronald L. Rivest
Len Adleman
Michael L. Dertouzos





Fully/somewhat homomorhic encryption

$$f\left(Ctxt(m)\right) = Ctxt(f'(m))$$

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More efficient in practice

Many useful functions are not arithmetic

- Trigonometric functions
- Sigmoid/step functions
- Comparison functions:
  - logical predicates “is equal”, “is less than”
  - $\max(x, y)$, $\min(x, y)$
  - $\text{argmax}(x_1, \dots, x_n), \text{argmin}(x_1, \dots, x_n)$

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Computation complexity in HE

Cheap operations	Expensive operations
Plaintext + ciphertext	Ciphertext * ciphertext
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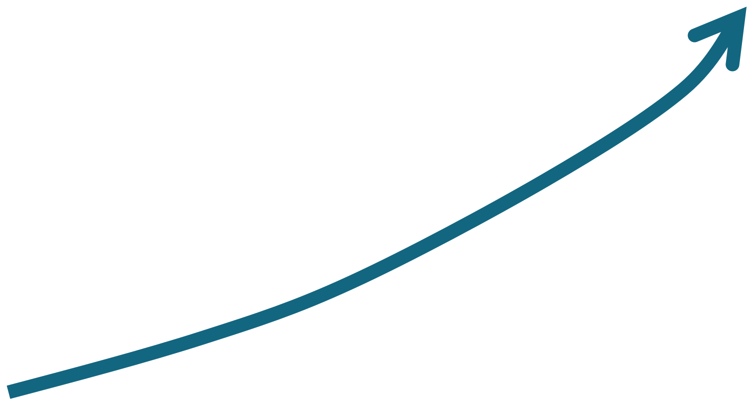
$P(ctxt) = a_0 + a_1 \cdot ctxt + a_2 \cdot ctxt^2 + \cdots + a_{n-1} \cdot ctxt^{n-1}$

Scalar multiplications are cheap, non-scalar ones are expensive.

Computation complexity in HE

Encryption parameters



Non-scalar multiplicative depth

Context

HE schemes

Arithmetic	HE schemes
Bit-wise	FHEW, TFHE
Integers	BGV, BFV
Approximate (fixed-point)	CKKS/HEAAN

HE schemes

Arithmetic	SHE/FHE schemes
Bit-wise	FHEW, TFHE
Integers	BGV, BFV
Approximate (fixed-point)	CKKS/HEAAN

BGV and BFV can

- evaluate arithmetic circuits
- encode data as elements of $\mathbb{F}_{p^d}$

Plaintext space

$$\mathbb{F}_{p^d}^{\ell}$$

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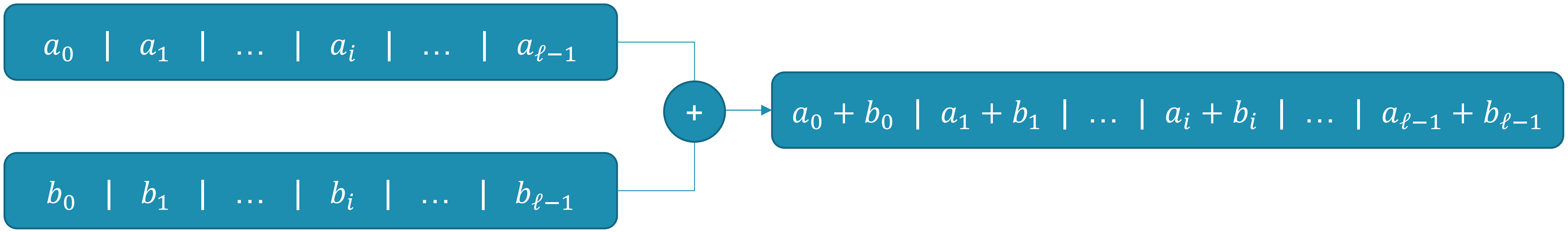

Parallel (SIMD) operations on ℓ slots!

Possibility to add, multiply, rotate, select the different slots.

Plaintext encoding of large integers

- Decompose an integer a in base $p' \leq p$: $a = \sum a_i p'^i$

a_0	a_1	$\ldots$	a_i	$\ldots$	a_{r-1}

Each a_i is also an element of $\mathbb{F}_p$

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- Every group of d digits can be mapped to an element of $\mathbb{F}_{p^d}$

$\mathbb{F}_p^d$ $\qquad\qquad$ $\mathbb{F}_{p^d}$

a_{id} | … | $a_{i(d+1)-1}$ $\longleftrightarrow$ $b_i = a_{id} + a_{id+1}X + \cdots a_{i(d+1)-1}X^{d-1}$

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$\mathbb{F}_p^r$ $\quad\longleftrightarrow\quad$ $\mathbb{F}_{p^d}^{r/d}$

$a_0 \mid a_1 \mid \ldots \mid a_{r-1}$ $\quad\longleftrightarrow\quad$ $b_0 \mid b_1 \mid \ldots \mid b_{r/d-1}$

Computations over $\mathbb{F}_p$

Equality function over $\mathbb{F}_p$:

$$\mathrm{EQ}_{\mathbb{F}_p}(x, y) = 1 - (x - y)^{p-1} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

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Lagrange Interpolation

Every function $f\colon \mathbb{F}_p^n \to \mathbb{F}_p$ can be interpolated by a unique polynomial of degree at most $p - 1$ in each variable

$$P_f(X_1, \ldots, X_n) = \sum_{\boldsymbol{a} \in \mathbb{F}_p^n} f(\boldsymbol{a}) \prod_{i=1}^{n} \mathrm{EQ}_{\mathbb{F}_p}(X_i, a_i)$$

Homomorphic comparison algorithm [TLW+20]

Input: two encrypted integers $\boldsymbol{x}$ and $\boldsymbol{y}$ encoded into $\mathbb{F}_{p^d}^{\ell}$

x_0	x_1	…	x_i	…	$x_{\ell-1}$

y_0	y_1	…	y_i	…	$y_{\ell-1}$

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$x_0 \mid x_1 \mid \ldots \mid x_i \mid \ldots \mid x_{\ell-1}$ $\quad$ $y_0 \mid y_1 \mid \ldots \mid y_i \mid \ldots \mid y_{\ell-1}$

1. Extract digits from $\mathbb{F}_p$

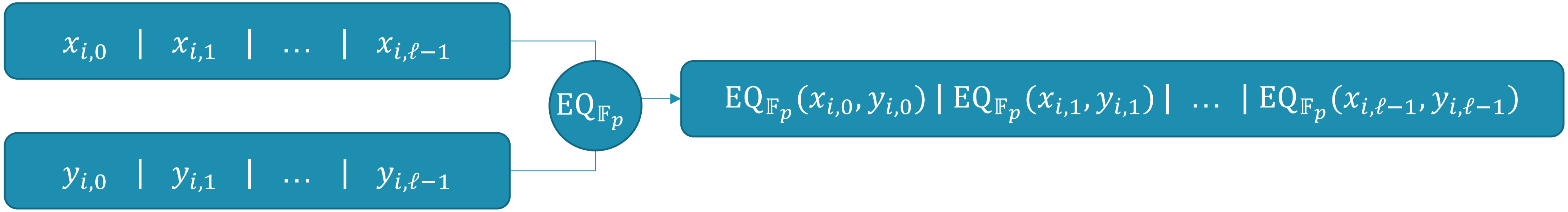
$x_{0,0} \mid x_{1,0} \mid \ldots \mid x_{\ell-1,0}$ $\quad$ $y_{0,0} \mid y_{1,0} \mid \ldots \mid y_{\ell-1,0}$

…

$x_{0,d-1} \mid x_{1,d-1} \mid \ldots \mid x_{\ell-1,d-1}$ $\quad$ $y_{0,d-1} \mid y_{1,d-1} \mid \ldots \mid y_{\ell-1,d-1}$

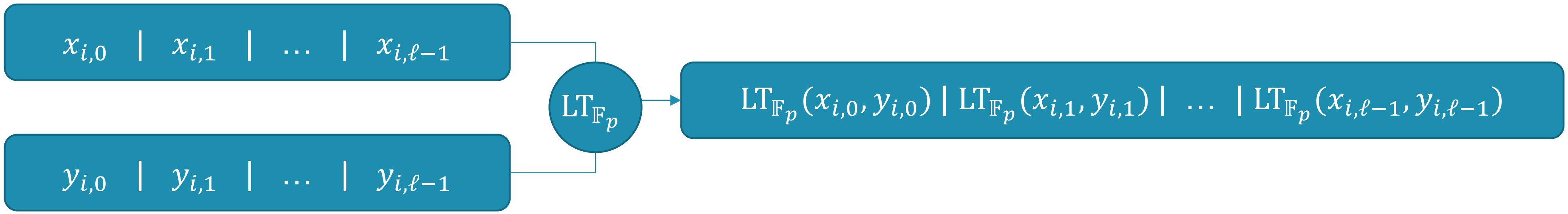
Homomorphic comparison algorithm [TLW+20]

2. Compare corresponding digits by computing the equality function



Homomorphic comparison algorithm [TLW+20]

3. Compare corresponding digits by computing the less-than function



$$\mathrm{LT}_{\mathbb{F}_p}(x, y) = \begin{cases} 1 \text{ if } \; x < y \\ 0 \text{ otherwise} \end{cases}$$

Homomorphic comparison algorithm [TLW+20]

4. Compute the lexicographical order

$$\mathrm{LT}_{\mathbb{F}_p^d}(x_i, y_i) = \sum_{j=0}^{d-1} \mathrm{LT}_{\mathbb{F}_p}\left(x_{j,i}, y_{j,i}\right) \prod_{k=j+1}^{d-1} \mathrm{EQ}_{\mathbb{F}_p}(x_{k,i}, y_{k,i}),$$

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Contributions

Core part of integer comparison

4. Compute the lexicographical order

$$\mathrm{LT}(x_i, y_i) = \sum_{j=0}^{d-1} \mathrm{LT}_{\mathbb{F}_p}(x_{j,i}, y_{j,i}) \prod_{k=j+1}^{d-1} \mathrm{EQ}_{\mathbb{F}_p}(x_{k,i}, y_{k,i}),$$

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How to compute $\mathrm{LT}_{\mathbb{F}_p}(x, y)$: bivariate method

Let $x, y \in [0, p - 1]$.

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x \ y	0	1	2	3	…	$p-1$
0	0	1	1	1	…	1
1	0	0	1	1	…	1
2	0	0	0	1	…	1
3	0	0	0	0	…	1
…	…	…	…	…	…	…
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$$P_{\mathrm{LT}_{\mathbb{F}_p}}(X, Y) = \sum_{a=0}^{p-2} \mathrm{EQ}_{\mathbb{F}_p}(X, a) \sum_{b=a+1}^{p-1} \mathrm{EQ}_{\mathbb{F}_p}(Y, b)$$

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$3p - 5$ non-scalar multiplications [TLW+20]

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$$\mathrm{LT}_{\mathbb{F}_p}(x, y) = \mathrm{IsNegative}_{\mathbb{F}_p}(x - y, 0)$$

$x - y$	$-\frac{p-1}{2}$	…	-3	-2	-1	0	1	2	3	…	$\frac{p-1}{2}$
$\mathrm{IsNegative}_{\mathbb{F}_p}$	1	…	1	1	1	0	0	0	0	…	0

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$\sqrt{2p - 2} + \mathcal{O}(\log p)$ non-scalar multiplications [PS73,SFR20]

Bivariate method

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Our results:

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$$f(X,Y) = \sum_{i=0}^{(p-3)/2} f_i(X)Z^i$$

with $Z = Y(X-Y)$ and $\deg f_i(X) = p-3-2i$

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Non-scalar multiplications:

$2p-6 < 3p-5$ [TLW+20]

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  $\Longrightarrow$ save $\mathcal{O}((d-1)\log p)$ non-scalar multiplications for the lexicographical order!

Min/max

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- use the univariate approach

$$Q_{\min}(X, Y) = \frac{p+1}{2}(X + Y)^{p-1} + g((X - Y)^2)$$

- Saves one multiplicative level.
- Same complexity as for evaluating $\mathrm{LT}_{\mathbb{F}_p}$
- Similar method can be applied to evaluate $\mathrm{ReLU}(x) = \max(x, 0)$

Sorting [CDS+15]

Let $A = [a_0, a_1, \dots, a_{N-1}]$ be an array of numbers

$$A = [5,1,7,2]$$

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$$\boldsymbol{L} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Complexity: $N(N-1)/2$ homomorphic comparisons

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Positions in the sorted array

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$$M = [2,0,3,1]$$

2. Select element with index i in the sorted array A_{sorted}

$$\mathrm{EQ}_{\mathbb{F}_p}(M[j], i) \cdot a_j = \begin{cases} a_j \text{ if } M[j] = i \\ 0 \text{ otherwise} \end{cases}$$

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$$A_{sorted} = [a_1, a_3, a_0, a_2] = [1,2,5,7]$$

Min/max element of array

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- Use the sorting algorithm $\Longrightarrow N(N-1)/2$ homomorphic comparisons ✘

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- Use the sorting algorithm $\Longrightarrow N(N-1)/2$ homomorphic comparisons ✘
- $N-1$ successive comparisons $\Longrightarrow$ depth too big ✘

Min/max element of array

Let $A = [a_0, a_1, \dots, a_{N-1}]$ be an array of numbers

Use the tournament method to mix both strategies an obtain the best trade-off

a_0

a_1

a_2

a_3

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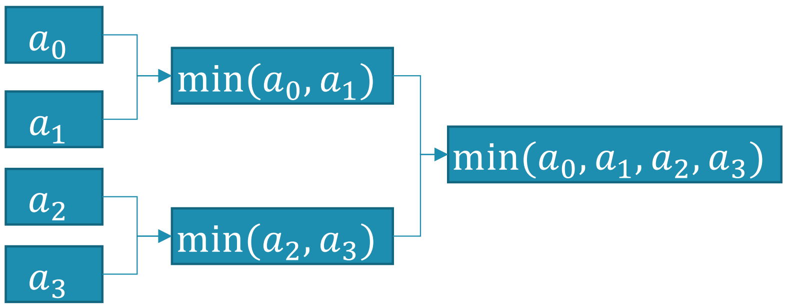

- Use T stages of the tournament method

Min/max element of array

Let $A = [a_0, a_1, \dots, a_{N-1}]$ be an array of numbers

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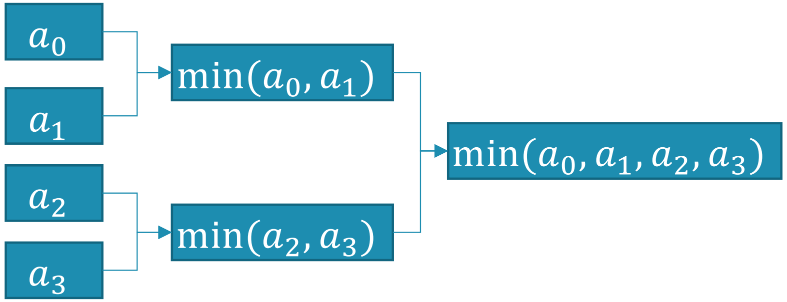

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  Only need to sort $N' = N/2^T$ elements

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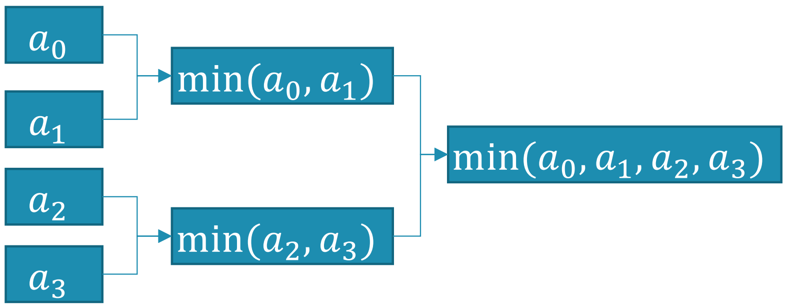

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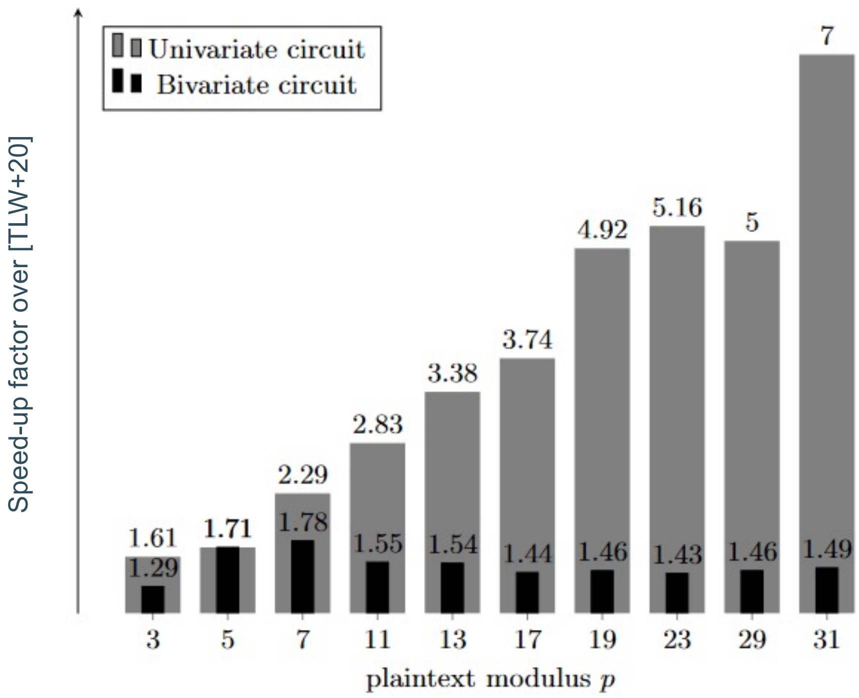
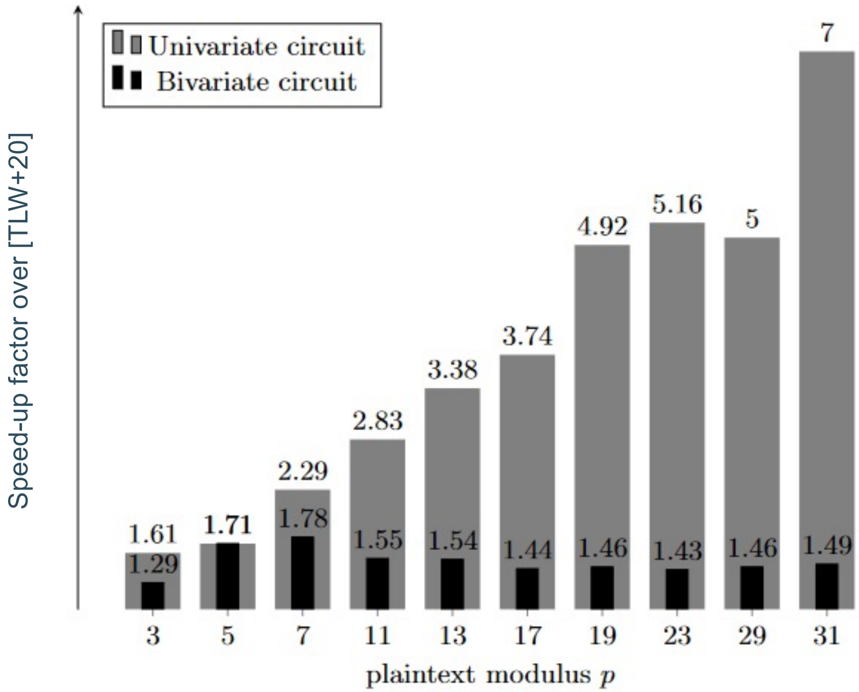
Complexity: $(N - N')$ min + $N'(N' - 1)/2$ less-than functions

Implementation

Less-than function for 64-bits integers



Less-than function for 64-bits integers



Best running time

	Prior work (TLW+20)	This work
p	5	131
Total	24.97s	16.07s
Amortized per integer	36ms	11ms

Sorting N 32-bits integers

N	Total time (s)	Amortized time (ms)	Amortized time [CDS+15]
4	299	64	200
8	1,356	290	944
16	5,700	1,219	4,280
32	23,017	4,922	18,600
64	89,972	19,241	49,700

Time to sort N 32-bits integers with 92 bits of security

Minimum of N 32-bits integers

N	Total time (s)	Amortized time (ms)
2	38	15
4	158	60
8	506	194
16	1,694	649
32	6,440	2,467
64	24,986	9,573

Time to extract the minimum of N 32-bits integers with 121 bits of security

Comparison with other FHE schemes

Bit length	FHE scheme	Bits of Security	Total time (s)	Amortized time (ms)
12	TFHE* CKKS BGV	156 128 126	0.002 127.5 7.09	2.04 1.95 1.23
16	TFHE* CKKS BGV	156 128 126	0.003 297.0 12.11	2.72 4.53 2.10
20	TFHE* CKKS BGV	156 128 126	0.003 373.8 8.66	3.40 5.70 3.01

Timings for the less-than function

* TFHE timings are estimated from [CGG+20]

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Future work: other useful functions over rings/fields with effiicient circuits?

Thank you!